Tamil Nadu Curricular Framework 2017
A statement on Mathematics Curriculum

1 Goals of math education

What is mathematics about ? In school, the content areas of mathematics are, broadly: arithmetic, algebra, geometry, trigonometry, mensuration and data analysis. All these areas bring about certain calculational skills and factual knowledge in the learning child, and these are indispensable for the learning of sciences, as well as for certain everyday tasks one meets in life. But every educator agrees that these do not constitute the central purpose of mathematics education. As George Polya observes ([Po69]), if mathematics is about anything at all, it is about a certain way of *thinking*. Its uniqueness lies in training the mind to think abstractly, make connections between and perceive patterns in abstractions, and out of these build a coherent logical body of knowledge articulated in precise and reliable form, one that relies on a rigorous standard of proof. It speaks in the language of numbers and formulas, but doing mathematics is about thinking in a certain way, and communicating thought in a certain way, one that humanity has found to be remarkably successful over centuries. For this purpose, the *process* of mathematics is (arguably) more important than its products ([Fr]).

In this sense, the central goal of mathematics education is *to mathematize thought* in the learning child. According to David Wheeler ([W82]), it is more important to be able to think mathematically than know a lot of mathematics. The content areas of mathematics then become vehicles to promote thought and communication rather than ends in themselves. The growing child develops an understanding of the natural world and society, not only in the present, but in its past and potential future as well. So also does the child develop facility in the inner world of thought, abstraction and mental representations of patterns observed in the world. If natural languages provide the best medium of communicating our understanding in all these realms, formal (mathematical) language provides an excellent medium of thinking and communicating abstractions and representations. The famous assertion that the sum of all three angles of a triangle equals two right angles is one such communication, composed entirely of abstractions and representations, and can be understood and defended entirely in such a language. If any single reason can be isolated for the remarkable success mathematization has led to in many fields of thought, it is this ability to build layers of abstraction using rigorous tools.

The main aim of mathematics education, then, is to provide educational experiences in school that enables the child to learn this language, to gain fluency with its modes of expression, to think and communicate in such a fashion. For instance, someone who has completed schooling and drives an autorickshaw for a living, should be able to speak of what he earns on average, reflect on variations in income, consider what changes would be needed on a daily basis if the monthly income were to increase by half, and discuss the relative desirability of long distance and short distance rides. All of this involves some calculational ability for sure, but in the absence of *mathematical thought*, the driver would never make the calculational effort at all, and would very likely be unable to take charge of his practice in a professional manner. A similar remark would apply to a majority of the millions of self-employed in the country. Thus the goal of mathematics education is not to provide computational skills (that calculators can accomplish better) or informational knowledge (that search engines can deliver easily), but to influence citizens' thought processes in such a way that society can manage its resources efficiently and equitably.

How is this to be done ? The content areas of mathematics provide plenty of opportunity for the child to train the mind to think logically, abstractly, critically and creatively. Classroom processes need to improve the child's ability to mathematically articulate, analyze and solve meaningful problems. Textbooks and other educational material need to enhance the child's ability to make rich connections across mathematical ideas ([T90]). At the heart of all this is developing a healthy pre-disposition towards *problem solving*, one that nurtures interest and instils confidence in the child. In school, mathematics is the principal domain for formal problem solving, and it is an invaluable life skill that it offers. Perhaps the best way to gain fluency in mathematics is to learn to enjoy the challenges of problem solving and develop means for addressing them.

Tamil culture justifiably takes pride in a tremendous mathematical heritage. Tamil is an ancient language that delights in coining expressions such as *mukkaale araikkaal veesam* (3/4th of 1/8th of 1/16ths !). Tamil society has constituted practices such as **Nellilakkanam** (arithmetic of grain: the large) and **Ponnilakkanam** (arithmetic of gold: the miniscule), and composed texts such as **Kanakkadhigaaram**. The payol schools (*thinnai pallikkoodangal*) of Tamil Nadu had an elaborate system of arithmetic education long before Europeans set foot on this soil. Tamil sculptors are famous for their calculations of proportion and their geometric insights. All this heritage attests to the centrality this culture has always accorded to mathematical knowledge and education. However, every era has to redefine its goals, retaining such continuity, while looking firmly at the future and maintaining centrality of purpose.

Perhaps the most romantic figure in the world history of mathematics is that of **Srinivasa Ramanujan** and it is this society that produced such a genius. If we can communicate a tiny fraction of Ramanujan's enjoyment of mathematics to every child, we will transform our society: for, enjoyable mathematics leads to applications that nobody can foresee. Ramanujan's own mathematics, conceived without any idea of applications, leads to amazing applications to this day.

2 The central problems

Any recommendation for curricular action needs to be based on what it perceives to be the central problems of the extant system. Since there are many potential problem areas in all education, identifying the ones of central concern is important. In this sense, we point to the following areas of concern in school mathematics:

1. **Fear and alienation**: A majority of school children express a certain fear of mathematics, or perceive a certain alienation from mathematical activity in school. This prevents their engagement in mathematics and over time, teachers do not expect their engagement.

2. **Reliance on rote learning**: Excessive emphasis on performance on examinations has led to all problem solving being reduced to learning the given solutions of a limited list of problems, sustained by the practice of asking only those problems *verbatim* in exams. This leads to rote learning, entirely contrary to the spirit of mathematical thinking.

3. **Inadequate teacher preparation**: Teacher education curricula rarely deal with content knowledge of mathematics, and university mathematics is entirely different from school mathematics. Thus teachers have no opportunity to critically examine content and pedagogy of school mathematics, and hence are unable to flexibly generate content as needed.

It is worth emphasizing that these concerns are listed in a formulaic manner and need substantial elaboration.

All of this together lead to a substantial problem: a significant number of children barely acquire sufficient mathematical skill and understanding despite years of compulsory schooling in mathematics. The objectives of mathematical learning, as outlined above, remain unachieved for most.

There are other **systemic** problems as well, relating to axes of social discrimination like caste, class and gender. For now, we merely note that they exist, leaving it for discussion elsewhere.

3 The main recommendations

We recommend the following steps for action, as a means not only for addressing the problems articulated above, but also for progress towards realising the vision we seek. These are broadly in line with those of the National Curriculum Framework 2005 ([NFG]).

1. Ensuring the attentive engagement of **all** children in mathematics with a sense of enjoyment and success,
2. Shifting the focus of mathematics classroom from content knowledge and skills to a variety of processes,
3. Changing assessment models to shift away from rote learning and encourage mathematical thinking through active problem solving, and
4. Enriching teachers with a variety of mathematical resources.

The first one involves a commitment to forms such as activities, puzzles and games that encourage interested participation by all children, and group explorations that help children work together, and enhance each others' understanding. The shifting of focus advocated here involves realisation of many processes in the classroom such as visualisation, use of heuristics, estimation and approximation, optimization, use of patterns, use of multiple representations, reasoning and proof, making connections, mathematical communication, and so on. According to NCF 2005, making room for such processes constitutes the difference between "doing mathematics and swallowing mathematics, between mathematization of thinking and memorizing formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims" ([NFG]).

None of this can be achieved without a significant change in assessment modes, those that give an experience of wide-ranging problem solving. Realising these requires empowering teachers by providing them with a range of educational resources.

4 The role of technology

There are many areas of mathematics education where the use of technology may be a distraction; many where it can significantly enhance learning; and many where it is indispensable and we are currently missing out on a valuable educational resource. Identifying and demarcating these areas is the contemporary challenge and doing so will be critical for addressing the needs of the future.

Firstly, by technology, we do not necessarily mean Information and Communication Technology (ICT). The humble geometry box has provided excellent support for generations of children, and the compass or graph sheets are still very good tools. For children in the primary school, Jodo Gyan kits, Lego blocks and other such material are immensely useful. The point here is the imaginative use of such technology for engaging all children in mathematical activity.

Secondly, even when it comes to the ICT, we are not speaking of "smart classrooms" where videos are downloaded and shown to children. Videos can indeed be useful, and they have their place, but it is a marginal place.

Instead imagine a child coming across a linear equation, clicks on it to get a window with the graph of the function. A question asks what happens if the x coefficient is doubled ? The child makes a guess, then verifies by changing the coefficient and seeing the new graph. Better, she starts and pulling and stretching the curve, her eye on how the equation keeps changing. This is the kind of interactive experience where technology can greatly enhance learning.

Consider the processes such as visualisation, use of patterns, estimation and approximation, making connections, referred to above. The potential of technology for mathematical engagement in such processes is tremendous. To take a simple example, when small children learn the number line, it is very hard for them to understand that it looks the same whether we are looking at the segment from 31 to 40 or from 2151 to 2160; and yet a simple 'game' on a mobile phone that lets you pick up one segment and impose it exactly on the other shows how well they match and give an intuitive feel. Similarly when it comes to negative numbers during the upper primary stage or exploring how rationals are dense (by zooming in further and further to see that the process never stops). The role of technology in geometry and graphing functions is yet again immense, as even experts cannot easily tell how changes in coefficients of polynomials affect the shape of the curves they represent, whereas a simple tool can show it visually. Such open ended explorations can not only engage children's interest but also enhance their ability to think abstractly.

While there are many substantial issues of access and equity when we consider providing all children with such technology, not to forget its addictive nature as well, what we are emphasizing here is the tremendous expansion of resources that it provides to our classrooms, in terms of their own enrichment and ability to flexibly and dynamically generate content as required. For children, shared classroom access to technology is sufficient for accessing this potential, at least for now.

In teacher empowerment technology can play a major role today, one that was perhaps not available to any earlier generation.

However, a serious re-orientation of curriculum, pedagogy and assessment is required for such integration of technology into the school mathematics classroom.

5 Stage-wise curricula

There are certain themes that run through all the stages of mathematical learning. Number and shape are the central concepts of the primary school, and they are perceived independently, bringing order and method into their conception and use. The process of arithmetization, attaching quantities to not only physical entities (for measurement) but also to shapes (for extent) begins at the primary school but gains solidity as science is learnt. Algebra, introduced as generalized arithmetic at the upper primary level, offers a language for talking of number, shape and almost anything "mathematical". Almost all elementary mathematics, directly applied in every day life, for much of economic life and aesthetics, is expressible in the language of arithmetic, algebra and geometry learnt at this stage. Fluency in this language is not computational skill that would be easily achieved with a calculator, but an ability to think quantitatively, perform mental arithmetic so as to determine need and estimate quantities accordingly.

At the primary stage, it is the progression from the concrete to the abstract that is central, and it needs emphasis that mathematics is not 'just' arithmetic. This means that even while addressing number and number operations, due place is given to shapes, spatial understanding, patterns, measurement and data handling. It is acknowledged the world over that fractions and decimals constitute a major problem area at the primary stage and lead to fear of mathematics among many children. Hence the content in these areas needs careful consideration.

The upper primary stage is one of consolidation, a reinforcement of whatever is learnt earlier in abstract and compressed form, easily usable later on. With a new mathematical language and extensive introduction to formal problem solving and abstract reasoning, learning at this stage has to grapple with the essential components of mathematical thinking. Enjoyment of mathematics at this stage is strongly correlated with confident engagement in science and mathematics during all the school years.

In secondary school, mathematics emerges as a discipline, and fluency in the language of mathematics becomes essential for understanding, as all concepts are articulated in the language of algebra. Number, shape and indeed every concept, is represented in a way that one can go back and forth betweeen these representations. The best example of this is in coordinate geometry: a curve, expressed in algebraic form, is plotted on graph paper using numbers, to generate a shape. A change in the shape then leads to a new equation, arrived at using numerical coordinates again. This stage involves a transition from additive to multiplicative reasoning, and from inductive to deductive reasoning.

While the infinite is intuitively grasped in high school mathematics, and used extensively (in science, especially physics) by way of real variables, it is at the higher secondary level that it takes centre-stage. This is in some sense the onset of disciplinary mathematics, with the introduction of infinite sums, limits and the differential calculus, as also trigonometric functions, loci of points, real random variables etc. The power of mathematics manifests in treating infinite objects as manipulable and calculable and at this stage, this becomes the central facility.

Note the cyclic repetition of themes across stages. Whole numbers provide approximations to rationals which in turn approximate the reals; regular shapes such as triangles and rectangles provide approximations to quadrilaterals (and many irregular ones met with in nature). Notions of ratio and proportion, scaling and placing bounds on change are central, and this is studied through multiple means across the stages. While commercial mathematics or heights and distances may illustrate the use of such thinking in everyday life, the concept itself goes well beyond, and leads to trigonometry later on. Visualization and geometric representations, forming equations and solving them, forming arguments and understanding abstractions via formal definitions – these processes run through the content areas all along from classes 6 to 12.

An important source of tension to refer to here is the potential conflict between procedural skill and conceptual understanding. Excessive reliance on one to the exclusion of the other is problematic in mathematics. Perhaps one can draw a parallel between reading and mathematics here. When one is reading letters and words it is hard to comprehend meaning, and it is only with some speed in reading, with anticipation and guesswork (on encountering unfamiliar words), that the process of reading ceases to be mechanical and meaning "comes automatically". So also with the language of arithmetic and algebra: fluency in operations and equation solving greatly helps in anticipating, seeing patterns and meaning making. On the other hand, emphasis on procedure without accompanying understanding can translate to computational ritual and fragile learning that cannot withstand even small changes.

5.1 Two new dimensions

While the foregoing discussion is based on a critique of extant curriculum and consideration of what needs to be done to make mathematics education meaningful and enjoyable, we recommend two new dimensions with an outlook for the future, for mathematics learning in school over the next decade. These are: **mathematical modelling** and **mathematics for the digital era**. We briefly describe them, the curricula will need to evolve based on experimentation and learning from experience.

5.1.1 Modelling

While mathematical modelling has always been central to the use of mathematics in various professions, it has remained marginal in our curricula. Most educators have agreed on the importance of modelling, both from the perspective of its usefulness for the student, but also from a pedagogical point of view, for its intrinsic mathematical value. However, limitations of textbook presentation and assessment have kept modelling in the margins of school curricula.

With the powerful technological tools available today and likely to only get better in the near future, the possibilities for integrating modelling into school curriculum are multiplied manifold. Modelling requires playing around with variables to decide which ones are relevant, with alternative visualisations to consider which description fits best and forming (in)equations rather than solving them. The use of technology offers new opportunities in this regard that did not exist earlier. For instance, a student can manipulate a rotating wheel on a mobile screen dynamically, experimenting with its radius and speed, observing various properties (even choosing between the desired ones) to select whatever is needed for a particular model. At different stages of school, modelling can involve physical material and virtual ones. At the primary stage, it is important to make clay models of three dimensional objects; at the upper primary, mechanical models (with dynamics) are useful; later on, computing technology can play a significant role.

Most educators consider that the twenty first century workplace will greatly demand modelling, interactively making use of information and computation, with students working together in project mode. There are rich modelling possibilities appropriate to every stage of school, not only using the mathematical knowledge and skill available at that stage, but also enhancing them powerfully.

5.1.2 Mathematics for the digital era

It is almost a cliché to talk of the ubiquitousness of computers and Internet in modern life. Algorithms are taking over the running of many aspects of everyday life of the citizen, and understanding the world is going to increasingly involve understanding of its digital manifestations. Moreover a strong foundation for computational thinking will be essential for children growing up in this century. As it happens, such understanding and thinking lies squarely within the realm of mathematics in school.

We can envisage the possibilites along five lines:

- Systematic listing, counting, and reasoning,
- Modelling and organization of information,
- Iterative (repetitive) patterns and processes,
- Following and devising lists of instructions (algorithms), and comparing them for efficiency.

Each of these themes offers a range of activities and educational opportunities across the stages. For instance, systematic listing at the primary school might involve making different garlands with some number of yellow flowers and white ones. In middle school, it might mean listing the number of different 'words' that can be made using five letters. Later it is about determining the number of possible orderings of an arbitrary number of objects.

One can list possibilities similarly for all these themes. What we wish to emphasize is that such reasoning underlies the science of data and information organization, and getting our children tuned to such thinking at once expands their mathematical abilities and prepares them better for the digital era. This is important for our children to eventually contribute to the information revolution, and not grow up only as its consumers.

6 A vision for math education

In our vision, school is a place where:

- Children learn to *enjoy mathematics*.
- Children pose and solve *meaningful problems*.
- Children understand the *basic structure of mathematics*.
- Children learn to argue and pursue assumptions to *logical conclusions*.
- Children relate mathematics to *life experiences*, and talk mathematics.
- Teachers expect to *engage every child* in class.

7 Road map to implementation

To realise this vision, we need to act on several fronts as a society, over a period of time. The most critical dimensions on which action has to be taken up are listed below.

7.1 Board and school examinations, classroom assessment

Perhaps the most critical reform needed in mathematics education in Tamil Nadu is in the nature of Board Examinations for students completing 10 and 12. When examiners ask problems *verbatim* from so-called blueprints, students can simply rote-learn solutions to these problems and score maximum marks in the exam. This leads to a happy equilibrium between exam demands, student preparation to meet them and teacher effort at coaching for it, with no incentive for concept learning. With Board examinations casting long shadows, the culture of rote learning percolates all the way down, with all school exams and classroom assessment ending up as memory recall exercises.

Hence the first change has to be in assessment. While the minimum required for passing Board exams at Class 10 should be liberal, based on requirements of completing compulsory school education for citizenship, it should be mathematical competence, understanding and problem solving ability determining the upper end. Indeed, all assessment in mathematics should move towards becoming meaningful problem solving opportunities that enhance learning.

We recommend the following steps in this regard:

1. Within the next year, ensure that all tests and examinations have roughly a quarter of questions from outside the textbooks.

2. Over the years, gradually increase this component, so that rote learning is done away with. One way is to start providing necessary formulas and tables along with questions, so that applying them assumes importance.

3. At every level, a small set of problems should be challenging and non-routine, calling for making connections and combining concepts.

4. Over the years, we should gradually introduce and increase assessment of *process skills* such as ability to visualize, to abstract, to change representations, to search for counterexamples, to provide arguments etc.

This calls for the development of a major resource bank of **assessment models** and this needs to be taken up as a priority exercise.

7.2 Educational resources

The second major dimension is provision of additional learning resources to children. Firstly, the textbooks should be designed more imaginatively, with motivation and context, cross linking content across chapters and across disciplines, and puzzlers distributed throughout. Certain urgent steps needed are:

1. Every chapter of every book should have a *Challenge* section, providing material that is not necessarily needed by everyone.

2. Problems should be provided with gradations such as easy, direct application and hard.

3. All textbooks should be online accessible with clickable links to resources.

4. From Class 6 onwards, all textbooks should be integrated with (open source) mathematical software so that children become adept at using them for learning and exploration, creating new content for themselves.

Moreover, textbooks cannot be the sole source of mathematical knowledge. The following varieties of material need to be generated, and much of this can be done in the form of online resources, with a view to the future.

1. Supplementary texts on specific mathematical themes (for instance, on π), history of concepts, lives of mathematicians, people's mathematical practices, etc.

2. Problem books, with graded collections at different levels.

3. Puzzles, games and activities, at various levels.

4. Videos and animations (e.g tessellations).

5. Interactive applets for exploring a variety of themes (e.g. limits from either side).

Generating such a resource base is well within the intellectual capability of a state like Tamil Nadu, with its rich mathematical legacy and institutions of excellence in mathematics. What is needed is a commitment to resource generation and its extensive dissemination.

7.3 Teacher preparation and support

One major lacuna in the system is the serious inadequacy of a large proportion of teachers in Pedagogical Content Knowledge (PCK) of mathematics. School mathematics is almost entirely driven by pedagogy and child psychology at the primary level and almost entirely driven by content at the secondary level. Unfortunately, there is a tremendous discontinuity between school mathematics and university mathematics, and hence those who return to school (after university) to teach, suffer a double discontinuity. It is not merely a matter of knowing mathematics but one of revisiting content from a pedagogical perspective, keeping in mind children's learning difficulties.

Educationists term this "Pedagogical Content" and mathematics education seems to suffer extraordinaly in this regard. For instance, when the idea of infinite sum is introduced, what is important, what is desirable but not necessary, what is inessential ? How are these to be decided ? How are students' difficulties in understanding infinite sums to be identified and addressed ? (In some sense, a student who does not grasp infinite sums and get fluency in it is doomed in the arena of university mathematics or engineering mathematics or higher physics. But if it is so important, does it occupy an important place in pedagogy ?)

All this suggests that the school textbooks and teachers' own higher education in mathematics are, in themselves, inadequate to help teachers when it comes to high school mathematics. On the other hand, at the elementary stage, teachers need extensive innovative material that address children imaginatively, and here again teachers tend to be ill equipped.

It is also to be noted that teacher education curricula in the state have very little mathematical content or pedagogy. Except for a miniscule minority who specialize in mathematics pedagogy. we have largely a generalist workforce.

This calls for certain steps to be taken up:

1. Every textbook should be accompanied by a teachers' guidebook offering teachers much more content.

2. A mathematics teachers' network and portal needs to be set up for sharing and learning from others' experiences.

3. Teacher professional development should include pathways of learning mathematics from a pedagogic perspective, accumulating credits over the years leading to diplomas and degrees.

4. It is essential to break down the rigid compartments between primary school teachers and those at the secondary and tertiary stage, with systemic means for visits and resource generation.

All such steps are well within the realm of the do-able. Quality mathematics education is not only the right of every child, empowering our children for the 21^{st} century demands a firm commitment to quality mathematics education.

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